A NOTE ON A SINGLE VEHICLE AND ONE DESTINATION ROUTING PROBLEM AND ITS GAME-THEORETIC MODELS

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Abstract: In the paper a shortest-time routing problem is considered. A decision- maker has to choose one of several possible routes for a vehicle which should reach its destination as soon as possible. The time required to reach the destination depends on the chosen route and the state of traffic flow in the region. Under the assumption that the latter can be reasonably categorized into finite number of states game-theoretic models of the problem are proposed. Some examples and practical questions are discussed as well

Keywords: vehicle routing problem, shortest-time criterion, game against nature.

1. Introduction

The transportation industry facilitates the movement of goods for the purposes of trade, production and consumption. The transportation systems are expected to satisfy several quality factors, such as cost, time of the service and many others. To meet various expectations connected with transportations decision makers have to face many different problems in a number of different situations. One of the oldest and most famous problems in the area is the so-called Traveling Salesman Problem (TSP) which in terminology of the graph theory can be expressed as follows: given a set of *n* vertices (interpreted as customers) and weights for each pair of vertices (interpreted as cost of the travel), find a roundtrip of minimal total weight visiting each vertex exactly once. Recent developments and some interesting variants of TSP are presented in [1.]. Another class of problems in which customers are visited by a *number* of vehicles are called the Vehicle Routing Problem (VRP). As a matter of fact the VRP is a generalization of the TSP. The problem is to design routes for the vehicles so as to meet given constraints and objectives minimizing a given objective function (total cost, time, length of the route etc). The VRP has been an especially active and fruitful area of research over last decade. There have been numerous technological advances and new concepts that are of considerable interest to researchers and decision – makers. Now the Vehicle Routing Problem is the core of logistics distribution. The state of the art is presented and discussed in the book [6]. Some new ideas for modelling and solving vehicle routing problems are connected with the application of the game theory methods. Such approach to various conflict and cooperative situations arising in the VRP and other transportation problems can be found in papers [2., 3., 4., 5., 8., 9., 10.]

In our paper we consider a routing problem where a decision- maker has to choose one of several possible routes for a single vehicle which should reach its destination as soon as possible. We assume that the time required to reach the destination depends on the chosen route and the state of traffic flow in the region. We also assume that, according to decision maker knowledge, the obstacles connected with the traffic can be reasonably categorized into finite number of significantly different cases which will be referred to as *states of nature*. In such a situation we formulate zero-sum game against nature as a model of the problem. Next we discuss various concepts of the solution of the game and their interpretations. In the last section we present an example and address some practical questions.

2. Problem statement and its game-theoretic model

Let v_0,v_1 denote the vertices: the depot (source) and the destination (customer), respectively. The two vertices are connected via n edges (routs) denoted $d_1, d_2,..,d_n$. So, we have parallel link network. The decision-maker has to choose one of the routs for the vehicle. The objective is the minimization of the transportation time, however the time required to reach the destination depends not only on the chosen route but also on the state of nature which can be one of the possible *variants* denoted $s_1, s_2,\dots,s_m$, respectively. The estimated time necessary to reach the goal via route d_i , i=1,2,…, n, when the s_j, j=1,2,…,m, is the true state of nature is denoted T_{ij}. We assume that the decision maker does not know which state of nature *is* true (or *will be* during the travel). Such situation can be considered as a game against nature. In the game theory terminology $S=\{s_1, s_2,\dots, s_m\}$ are called strategies for nature while the edges $D=\{d_1, d_2,..,d_n\}$ are called strategies of the decision maker. Such a game is usually denoted by $< S,D,\mathbf{T} >$, where the *loss* function $\mathbf{T}$ is given by the following m-by-n matrix

$$\begin{array}{cc} & \begin{array}{cccc} d_1 & d_2 & \dots & d_n \end{array} \\ \begin{array}{c} s_1 \\ s_2 \\ \dots \\ s_m \end{array} & \begin{bmatrix} T_{11} & T_{12} & \dots & T_{1n} \\ T_{21} & T_{22} & \dots & T_{2n} \\ \dots & \dots & \dots & \dots \\ T_{m1} & T_{m2} & \dots & T_{mn} \end{bmatrix} \end{array}$$

Traditionally the strategies of the nature can be identified with the rows and the strategies of the decision maker can be identified with the columns of the matrix **T**.

2.1. Solution concepts

In non trivial variants of the matrix **T** (and such are usually connected with the routing problem) there are no obvious choices for the decision maker. The best choice in the case where the state of nature is s_i may be the worst when true state is s_k. The game theory provides us with various concepts of possible solutions in such a case. First it propose to use so-called minimax strategies. The strategy d_{k*} (i.e. the route with the index k*) is called minimax (or safety) strategy if it satisfies the following condition, see [7.]:

$$\max_i T_{ik*} = \min_k \max_i T_{ik} \tag{1}$$

Using such a strategy the decision maker can guarantee himself that his loss does not exceed the value $V^* = \min_k \max_i T_{ik}$. The value V* is called the *upper value of the game*. So using the minimax strategy the decision maker assures that the maximum possible travel time (with respect to different states of nature) is as short as possible. In other words, when using the strategy he knows that the travel time will not exceed the value V*, whereas if he use other strategies he risks that the travel time will be longer.

In many situations it is possible however to propose better solution. It can be achieved by introducing so-called mixed strategies. From the mathematical point of view a *mixed strategy* is a probability distribution on the set D of the decision-maker strategies $d_1,..,d_n$ (which from now on will be referred to as *pure* strategies). So, the mixed strategy of the decision maker is a vector $\boldsymbol{\beta} = [\beta_1, \beta_2, \ldots, \beta_n]$ with nonnegative coefficients β_i, i=1,...,n, satisfying the condition:

$$\sum_{j=1}^{n} \beta_j = 1$$

Similarly we can introduce the mixed strategies for nature as vectors $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_m]$ with nonnegative α_i, i=1,...,m, satisfying

$$\sum_{j=1}^{m} \alpha_j = 1$$

Then the expected loss $\hat{\mathbf{T}}$ for the decision maker is defined as

$$\hat{\mathbf{T}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_i \cdot T_{ij} \cdot \beta_j$$

Such a game will be denoted $< S^*, D^*, \hat{\mathbf{T}} >$.

The minimax mixed strategy $\boldsymbol{\beta}^*$ for the game $< S^*, D^*, \hat{\mathbf{T}} >$ is defined by the following condition:

$$\max_{\boldsymbol{\alpha}} \hat{\mathbf{T}}(\boldsymbol{\alpha}, \boldsymbol{\beta}^*) = \min_{\boldsymbol{\beta}} \max_{\boldsymbol{\alpha}} \hat{\mathbf{T}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \tag{2}$$

It appears that the upper value $v^* = \min_{\boldsymbol{\beta}} \max_{\boldsymbol{\alpha}} \hat{\mathbf{T}}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ for the game $< S^*, D^*, \hat{\mathbf{T}} >$ is very often less than the upper value V^* for the game $< S, D, \mathbf{T} >$.

2.2. The game with travel time limit

In real world transportation problems we may also deal with another interesting problem. It concerns the situations where the vehicle does not have to reach its destination as soon as possible but it *cannot be late* with its cargo and that is what really matters (e.g. because the customer will be absent and additional, relatively large, cost will be generated). To model such a situation we assume that a given travel time limit t^* is known to the decision maker and both the decision maker and the customer will be satisfied if the vehicle travel time is less than t^* - if so, it will be called a *success*. In such a case it would be reasonable for the decision maker to choose a strategy which maximizes the probability of the *success*.

Such a problem can be represented by a game $< S, D, \mathbf{T}^{01} >$, where the matrix $\mathbf{T}^{01}$ is given as follows:

$$\mathbf{T}^{01} = \begin{bmatrix} \mathbf{1}(T_{11}) & \mathbf{1}(T_{12}) & \ldots & \mathbf{1}(T_{1n}) \\ \mathbf{1}(T_{21}) & \mathbf{1}(T_{22}) & \ldots & \mathbf{1}(T_{2n}) \\ \ldots & \ldots & \ldots & \ldots \\ \mathbf{1}(T_{m1}) & \mathbf{1}(T_{m2}) & \ldots & \mathbf{1}(T_{mn}) \end{bmatrix} \tag{3}$$

where

$$\mathbf{1}(T_{ij}) = \begin{cases} 1 & \text{if} \quad T_{ij} > t^* \\ 0 & \text{if} \quad T_{ij} \le t^* \end{cases}$$

Now we can adopt both above described concept of the solution of the game. Minimax mixed strategy for the game $< S^*, D^*, \hat{\mathbf{T}}^{01} >$ guarantees the decision maker highest probability of the success.
In the next section we present a numerical example illustrating the presented concepts of the solution.

3. Numerical example and final remarks

Let us consider a problem where the matrix **T** is as follows:

$$\mathbf{T} = \begin{bmatrix} 25 & 30 & 35 & 40 & 30 \\ 25 & 30 & 35 & 60 & 80 \\ 65 & 60 & 55 & 50 & 30 \\ 35 & 50 & 45 & 60 & 50 \\ 35 & 40 & 55 & 60 & 30 \\ 70 & 50 & 55 & 55 & 30 \\ 30 & 45 & 40 & 45 & 30 \\ 30 & 35 & 40 & 55 & 40 \\ 80 & 35 & 40 & 45 & 30 \end{bmatrix}$$

It follows from the game matrix that the decision maker has five different routes to choose between and he distinguishes nine significantly different states of the traffic flow which may occur during the travel and which influence the travel time. The states may be connected with the car accidents (their frequency) or some cultural or social events which are unpredictable to the decision maker. In the matrix the travel time is given in minutes. So, for example, if the second route is chosen and the third nature state is true then, according to the game matrix, the travel will last approximately T_{32}=60 min.
Minimax pure strategy. To solve the game in pure strategies we should find the maximum travel time for each route, i.e. the maximum in each column of the matrix **T.** We obtain the values 80,60,55, 60 and 80. Thus the minimax pure strategy is to choose the third route and the upper value V^*=55. It means that using the third strategy decision maker assures that the travel will not last longer than 55 minutes.
Minimax mixed strategy. To solve the game in mixed strategies one should adopt a computer software or built equivalent linear programming problem, see [7.]. For our game we obtain the following minimax mixed strategy: $(0,0,\frac{5}{7},0,\frac{2}{7})$ and the upper value $V^*=335/7 \cong 47{,}9$ min. It means that the third route should be chosen with the probability 5/7, the fifth with probability 2/7 and remaining routes should be omitted . From practical point of view the probabilities may be interpreted as frequencies of particular choices in the situation where the distance is covered many times (for example in every week – seven days - during two days the route number three should be chosen and during five days the fifth one.). If so, the decision maker assures that the maximum travel time does not exceed 47,9 minutes (on average). We see that the time is about seven minutes shorter then when using the pure minimax strategy.
Solution of the problem with travel time limit. Now let us consider our problem in the case where a travel time limit t^* is given. It means that the distance must be covered in time that does not exceeds the limit and if so, it is of no importance how many minutes it will take. To solve such a problem matrix $\mathbf{T}^{01}$ must be constructed. Let us assume that t^*=50. Then the matrix $\mathbf{T}^{01}$ is as follows, see Eq. 3:

$$\mathbf{T}^{01} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

To obtain the solution in mixed strategies we use the computer software and obtain the minimax mixed strategy in this case: $(0,\frac{1}{2},0,\frac{1}{2},0)$. The upper value for this game is v*=1/2 . So, the optimal strategy tell us that we should choose in turn the first and third route with equal frequency 50% (e.g. every second day) and we should not take into account the first , third and fifth ones. Such strategy assures that with probability 1/2 the vehicle reach the destination in time. It is interesting to observe that the strategies which should be chosen in previous problem, should be omitted in this case (namely routes number three and five)

Our example is very simple because of its illustrative character. In real world application the number of possible routes may be larger as well as the number of states of the nature. However in many situations it is possible to obtain the data which enable us to form the matrix **T.** Then the proposed approach may be adopted to obtain suitable solution of the transportation problem.

Sometimes it is also possible that the decision maker has some prior information about the probabilities of the particular states of the nature (e.g. frequencies they occur with). It would lead to Bayes , robust or empirical Bayes formulation of the problem.

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